SIMILARITY IN THE THREE-DIMENSIONAL PENETRATION PROBLEM OF SOLID BODIES INTO AN IDEAL INCOMPRESSIBLE FLUID

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We consider, in a linear setting, the three-dimensional problem of the initial stage of vertical penetration of blunt solid bodies into an ideal incompressible liquid.

Problems concerning the interaction of structures with a fluid arose in connection with the landing of hydroplanes [1]; these problems were later studied intensively in connection with other engineering questions (see [2-4] and the surveys given therein). These problems are very involved and, in general, can only be solved numerically. Cases which lend themselves to analytical treatment are of special significance for verification of the accuracy of the numerical schemes employed. As in other areas of mechanics [5], a large number of such cases are associated with the phenomenon of self-similarity.

Self-similar problems dealing with the penetration of solid bodies (cones or wedges) into a fluid were considered, in linear and nonlinear settings, in a variety of papers [3, 4, 6, 7]. In [6] it was shown that motion of the incompressible fluid during penetration into it of a cone is self-similar, providing the penetration velocity is a power of the time. It was noted in [1], in a study of penetration, that it is necessary to take into account the rise in the free surface of the fluid, which leads to an increase in the wetted surface of the body. In [8], in a linear setting, an attempt was made to obtain self-similar solutions for a problem involving penetration of an elliptic paraboloid into a fluid at constant velocity; moreover, no additional conditions (of Wagner or Karman—Pabst type [2]) were imposed on the boundary of the wetted region, a situation which did not guarantee uniqueness of the solution. In the present paper we show (in a linear setting, with rise of the free surface of the fluid taken into account) that the problem in question is self-similar for penetration velocities varying as a power of the time for a wide class of bodies in which the form of the surface may be described by a positive homogeneous function.

An exact analytical solution of the linear three-dimensional problem involving penetration into an incompressible fluid is known only for bodies of revolution [2, 9]. Although this case, generally speaking, is not self-similar, it does, however, retain certain features of self-similarity: at an arbitrary time, wetted regions are circles and are, consequently, similar to one another. In what follows, we identify a class of penetration problems in which, at each instant, the wetted region varies according to a similarity relationship. This class contains all cases for which three-dimensional analytical solutions are known. In particular, we obtain exact analytical solutions for the case in which the wetted region is an ellipse.

1. Statement of the Problem. We consider the problem of penetration of a solid body into an initially quiescent weightless ideal incompressible fluid. The fluid occupies the halfspace $x_3 \geqslant 0$. The velocity V(t) of the body is directed perpendicular to the plane $x_3 = 0$ forming the free surface of the fluid. We define the origin of a Cartesian coordinate system at the point of initial contact of the body with the free surface. Axis x_3 is directed into the depth of the fluid; axes x_1 and x_2 lie along the initial free surface.

We assume that the angle between the tangent to the body and the free surface is small throughout the time interval considered (blunt body assumption). In this case we can use Wagner's assumption [1] (see also [2]): penetration of the body may be replaced by flow over a continuously expanding flat disk, the rate of expansion of which is equal to the rate at which the width of the wetted surface of the body increases, while the flow rate is equal to the rate of penetration (the problem is solved in a linear setting, i.e., all equations are linearized and boundary conditions are carried over to the horizontal surface of the fluid, $x_3 = 0$). We neglect the effect of jet splash [7]. The flow in question is assumed to be potential.

Thus the problem concerning the penetration of a body, whose surface is defined by a function $f(x_1, x_2)$, may be reduced to the problem of determining the velocity potential

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$\Phi(\mathbf{x}, t)$ and wetted region G(t), satisfying the following conditions:

Laplace's equation

$$(\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2)\,\Phi(\mathbf{x}, t) = 0 \quad \forall (x_1, x_2, x_3) \in R^3_+; \tag{1.1}$$

boundary conditions on the plane $x_3 = 0$

$$\begin{gathered}\partial\Phi(x_1, x_2, 0, t)/\partial x_3 = V(t),\ (x_1, x_2) \in G(t),\\ \Phi(x_1, x_2, 0, t) = 0,\ (x_1, x_2) \in R^2|G(t);\end{gathered} \tag{1.2}$$

conditions at infinity

$$\begin{gathered}\Phi(\mathbf{x}, t) \to 0 \text{ and } \partial\Phi(\mathbf{x}, t)/\partial x_i \to 0 \text{ as } \sqrt{x_1^2 + x_2^2 + x_3^2} \to \infty,\\ i = 1, 2, 3.\end{gathered} \tag{1.3}$$

We assume that at time t = 0 the fluid is quiescent and that for t > 0 it is in motion.

The wetted region is obtained from a kinematic relationship due to Wagner connecting the rising motion of a free surface fluid particle with the motion of the body:

$$\begin{gathered}f(x_1^*(t), x_2^*(t)) = \int_0^t \left\{V(\tau) - \frac{\partial\Phi\left[x_1^*(t), x_2^*(t), 0, \tau\right]}{\partial x_3}\right\} d\tau,\\ (x_1^*(t), x_2^*(t)) \in \partial G(t)\end{gathered} \tag{1.4}$$

($\partial G(t)$ is the boundary of the open wetted region G(t)).

Pressure p in the fluid is determined from the linearized Cauchy–Lagrange equation [10]

$$p(\mathbf{x}, t) = -\rho\partial\Phi(\mathbf{x}, t)/\partial t \tag{1.5}$$

(ρ is the density fo the fluid); force P resisting penetration of the body is given by the expression

$$P(t) = \iint_{G(t)} p\, dx_1\, dx_2. \tag{1.6}$$

2. Identification of a Class of Self-Similar Solutions. We seek a solution of the penetration problem in the class of functions having the following property for arbitrary positive λ at each point $\mathbf{x}$:

$$\Phi\left(\lambda^{\alpha_1}x_1, \lambda^{\alpha_2}x_2, \lambda^{\alpha_3}x_3, \lambda^{\beta}t\right) = g(\lambda)\,\Phi(x_1, x_2, x_3, t). \tag{2.1}$$

Here α_1, α_2, α_3, β are weights for coordinates x_1, x_2, x_3 and time t, respectively. In particular, if $\alpha_1 = \alpha_2 = \alpha_3 = 1$, then for $\lambda = t^{-(1/\beta)}$, it follows from Eq. (2.1) that $\Phi(x_1, x_2, x_3, t) = g^{-1}[t^{-(1/\beta)}]\ \Phi(x_1/t^{1/\beta}, x_2/t^{1/\beta}, x_3/t^{1/\beta}, 1)$, i.e., the solution is self-similar.

Remark. If a function Φ of n variables satisfies the condition

$$\Phi\left(\lambda^{\alpha_1}x_1, \ldots, \lambda^{\alpha_n}x_n\right) = g(\lambda)\,\Phi(x_1, \ldots, x_n) \quad \forall\lambda > 0, \quad \forall\mathbf{x} \in R^n, \tag{2.2}$$

then

$$g(\lambda) = \lambda^k \ (k \text{ is a constant}). \tag{2.3}$$

Actually, if we write λ in the form $\lambda \equiv \lambda_1(\lambda/\lambda_1)$ $(\lambda_1 > 0)$, condition (2.2) then assumes the form

$$\Phi(\lambda^{\alpha}\mathbf{x}) = g(\lambda_1)\, g(\lambda/\lambda_1)\, \Phi(\mathbf{x}). \tag{2.4}$$

From Eqs. (2.2) and (2.4) it follows that $g(\lambda)/g(\lambda_1) = g(\lambda/\lambda_1)$. Following standard calculations from dimensionality theory [6], we obtain formula (2.3). It is known [11] that a function Φ, which satisfies condition (2.1) (with relation (2.3) taken into account), is called a quasihomogeneous function of degree k with weights $\alpha = (\alpha_1, \ldots, \alpha_n)$ for its variables. A search for a solution of problem (1.1)-(1.4) in the form of a quasi-homogeneous function allows us to identify conditions under which the problem in question is self-similar and leads us to the following theorem.

THEOREM 1. For the problem under investigation let the form of the penetrating body be defined by the function $x_3 = -f(x_1, x_2)$, (where $f(x_1, x_2)$ is a positive homogeneous function of degree d, $d \geqslant 1$), and let the penetration velocity be a power function of the form $V(t) = v_0 t^a$, $a \geqslant 0$. Then if the solution of problem (1.1)-(1.5) is given at time t_1 by $\Phi(\mathbf{x}, t_1)$ and $G(t_1)$, then at any other time t the solution of the problem is determined in accordance with the similarity relations

$$\Phi(\mathbf{x}, t) = \lambda^{-[a(d+1)+1]/(a+1)} \Phi(\lambda \mathbf{x}, t_1); \tag{2.5}$$

$$[(x_1, x_2) \in G(t)] \Leftrightarrow [(\lambda x_1, \lambda x_2) \in G(t_1)], \tag{2.6}$$

where $\lambda = (t/t_1)^{-(a+1)/d}$, i.e., $t_1 = \lambda^{d/(a+1)} t$.

Theorem 1 may be proved by a direct substitution of expressions (2.5) and (2.6) into Eqs. (1.1)-(1.5).

COROLLARY. Velocities of the fluid particles and pressure in the fluid at time t are determined from the formulas

$$v(\mathbf{x}, t) = \lambda^{-ad/(a+1)} v(\lambda \mathbf{x}, t_1), \quad p(\mathbf{x}, t) = \lambda^{-[a+1+d(a-1)]/(a+1)} p(\lambda \mathbf{x}, t_1). \tag{2.7}$$

The following qualitative results (valid under the assumptions made above) follow from Theorem 1: 1) the dimension ℓ of the wetted region varies directly with time to the power $(a + 1)/d$; 2) the area of the wetted region varies directly with time to the power $2(a + 1)/d$; 3) the force P, acting on the penetrating body from the fluid side, is directly proportional to time raised to the power $[(3a + 1) + (a - 1)d]/d$.

Actually, it follows from relation (2.6) that the size of the contact area is directly proportional to λ^{-1}, whence, substituting the value λ for $t_1 = 1$, we obtain the first result and, from it, the second, since the area of the region is proportional to the square of a linear dimension. The third result comes about from substituting relations (2.7) into Eq. (1.6) and integrating the resulting expression, taking relation (2.6) into account.

Remarks. 1. Results 1-3 for a = 0 and d = 1 were obtained in [6] from dimensionality theory. 2. Theorems analogous to Theorem 1 are valid for other media also: for example, in a contact problem of linear elasticity theory (in the isotropic case, see [12, 13]); in a contact problem involving contact of pre-deformed elastic halfspaces [14]; in dynamic contact problems for compressible media (an elastic halfspace, a compressible fluid halfspace). Here the exponent a in the penetraton velocity is inflexibly connected with the degree of homogeneity of the function defining the shape of the body: $a = d - 1$.

3. Solution of the Inverse Problem. We consider the problem inverse to the penetration problem, i.e., we are required to find the form of the penetrating body, given the velocity potential and the region of wetting. Validity of the following theorem can be readily verified directly.

THEOREM 2. Let us assume that in the halfspace $x_3 \geqslant 0$ we know a harmonic function F such that it and its gradient vanish at infinity and that in an open simply connected domain G(1), lying in the plane $x_3 = 0$ and containing the coordinate origin, the function F satisfies the conditions

$$\partial F(x_1, x_2, 0)/\partial x_3 = 1, \ (x_1, x_2) \in G(1), \ F(x_1, x_2, 0) = 0, \ (x_1, x_2) \in R^2 \backslash G(1). \tag{3.1}$$

Let $\varphi(t)$ be an arbitrary positive smooth monotonically decreasing function, where $\varphi \to \infty$ as $t \to 0$, and $\varphi(1) = 1$. Then the potential

$$\Phi(\mathbf{x}, t) = V(t) F(\varphi(t) \mathbf{x})/\varphi(t) \tag{3.2}$$

yields the solution of problem (1.1)-(1.4) describing penetration of a body into an ideal incompressible fluid with velocity V(t), where the function f_1 describing the form of the body is given by the expression

$$f_1(x_1^*, x_2^*) = \int_0^{t(x_1^*, x_2^*)} \left\{ V(\tau) \left[1 - \frac{\partial F(\varphi(\tau) x_1^*, \varphi(\tau) x_2^*, 0)}{\partial(\varphi(\tau) x_3)} \right] \right\} d\tau.$$

Here the wetted region G(t) is obtained from G(1) by a homothetic transformation with center at the coordinate origin: $(x_1, x_2) \in G(t) \Leftrightarrow (\varphi(t) x_1, \varphi(t) x_2) \in G(1)$. Here $t(x_1^*, x_2^*)$ is the time it takes the corresponding point of the boundary of the region G(1) to go into the point (x_1^*, x_2^*) lying on the boundary of the region G(t), i.e., $x_i^*(1) = x_i^*(t)\varphi(t),\ i = 1, 2$.

4. Solution of the Inverse Problem for an Elliptic Wetted Region. It is known [15] from the problem concerning flow over an elliptic plate by a potential flow that the function

$$F(x_1, x_2, x_3) = -\frac{(1-e^2)}{2E(e)} \int_{\omega}^{\infty} \frac{x_3 d\xi}{\xi^{3/2}(1+\xi)^{1/2}(1-e^2+\xi)^{1/2}} \tag{4.1}$$

is harmonic in the halfspace $x_3 \geqslant 0$ and vanishes at infinity. In addition, the function F satisfies conditions (3.1), where G(1) is an ellipse with semiaxes $a(1) = 1$, $b(1) = \sqrt{1 - e^2}$, and e is its eccentricity. Here $\omega(x_1, x_2, x_3)$ is a function which is equal to zero for $(x_1, x_2) \in G(1)$, $x_3 = 0$ and equal to the positive root of the equation

$$x_1^2/(1+\omega) + x_2^2/(1-e^2+\omega) + x_3^2/\omega = 1 \tag{4.2}$$

in the contrary case. Function E(e) is determined from the expression $E(e) = \int_0^{\pi/2} \sqrt{1-e^2\sin^2\xi}\, d\xi$.

If in Eq. (4.1) we calculate the derivative $\partial F/\partial x_3$ at $x_3 = 0$ and express the integral appearing in the resulting expression in terms of an elliptic integral, we have

$$\begin{gathered} \partial F(x_1, x_2, 0)/\partial x_3 = -\frac{1}{E(e)}\left[\frac{\sqrt{\omega+1-e^2}}{\sqrt{\omega(\omega+1)}} - E(e,\gamma)\right], \\ (x_1, x_2) \in R^2 \setminus G(1), \quad E(e,\gamma) = \int_0^{\gamma} \sqrt{1-e^2\sin^2\zeta}\, d\zeta, \quad \gamma = \arcsin\frac{1}{\sqrt{\omega+1}}. \end{gathered} \tag{4.3}$$

We find from Eq. (4.3) and Theorem 2 that the velocity potential

$$\Phi(x_1, x_2, x_3, t) = \frac{V(t)(1-e^2)}{\varphi(t)\, 2E(e)} \int_{\omega(\varphi(t)\mathbf{x})}^{\infty} \frac{\varphi(t) x_3 d\xi}{\xi^{3/2}(1+\xi)^{1/2}(1-e^2+\xi)^{1/2}}$$

gives the solution of the problem for a body penetrating at velocity V(t) (here the wetted region is an ellipse expanding homothetically according to the law $\varphi(t)$), the function defining the form of the body being given by the expression

$$f(r,\alpha) = \int_0^{t(r)} V(\tau)\left\{1 + \frac{1}{E(e)}\left[\frac{\sqrt{\omega(\varphi r, \alpha, 0) + 1 - e^2}}{\sqrt{\omega(\varphi r, \alpha, 0)\,[\omega(\varphi r, \alpha, 0) + 1]}} - E(e,\gamma)\right]\right\} d\tau. \tag{4.4}$$

Here we have used polar coordinates $r, \alpha\left(r = \sqrt{x_1^2 + x_2^2},\ x_1 = r\cos\alpha,\ x_2 = r\sin\alpha\right)$. We note, from Eq. (4.2) with $x_3 = 0$, that

$$\begin{gathered} \omega(\varphi r, \alpha, 0) = \frac{(\varphi r)^2 - (2-e^2) + \sqrt{e^4 + (\varphi r)^4 - 2e^2\cos 2\alpha\,(\varphi r)^2}}{2}, \\ \varphi r = \varphi(\tau)/\varphi(t). \end{gathered} \tag{4.5}$$

In particular, if we take a function describing the wetted region in the form $r(t) = \varphi^{-1}(t) = t^{(a+1)/d},\ a \geqslant 0,\ d \geqslant 1$, and the velocity V(t) in the form $V(t) = v_0 t^a$, we then find from Eq. (4.4) that the shape of the body may be described by a homogeneous function of degree d:

$$f(r,\alpha) = c(\alpha)\, r^d \tag{4.6}$$

$\left(c(\alpha) = v_0 \int_0^1 \xi^a \left\{1 + \frac{1}{E(e)}\left[\sqrt{\frac{\omega+1-e^2}{\omega(\omega+1)}} - E(e,\gamma(\xi))\right]\right\} d\xi\right.$ is a positive function of angle α). Here the function ω is given by expression (4.5) and $\gamma(\xi)$ is obtained from Eq. (4.3), in which it is necessary to put $\varphi r = \xi^{-(a+1)/d}$.

Remarks. 1. For a = 0, d = 2 we obtain the exact solution of a self-similar problem, in the Wagner formulation, for the case of penetration of a body into an incompressible fluid at constant velocity when the shape of the body is described by a homogeneous function of the second degree (4.6). 2. In the problem involving contact of a convex solid body with an isotropic elastic halfspace in the Hertz formulation (see, for example, [12, 13]) analogous to the penetration problem considered, it is assumed that the shape of an arbitrary body can be described by an elliptic paraboloid. Moreover, it is shown that the contact region is an ellipse. But if the problem of penetration of an elliptic paraboloid is considered in the Wagner formulation, the problem becomes a self-similar one (see Theorem 1); however, the region of interaction will be other than elliptical. 3. In the case of a circular region of interaction, the solution of a problem involving immersion of an arbitrary body of revolution in an incompressible fluid can always be obtained explicitly [2, 9].

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